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Generalized Hamiltonian formalism of (2+1)-dimensional non-linear σ-model in polynomial formulation

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Abstract. We investigate the canonical structure of the (2+1)-dimensional non-linear σ model in a *polynomial* formulation. A current density defined in the non-linear σ model is a vector field, which satisfies a *formal* flatness (or pure gauge) condition. It is the polynomial formulation in which the vector field is regarded as a dynamic variable on which the flatness condition is imposed as a constraint condition by introducing a Lagrange multiplier field. The model so formulated has *gauge symmetry* under a transformation of the Lagrange multiplier field. We construct the generalized Hamiltonian formalism of the model explicitly by using the Dirac method for constrained systems. We derive three types of the pre-gauge-fixing Hamiltonian systems: In the first system the current algebra is realized as the fundamental Dirac Brackets. The second one manifests the similar canonical structure as the Chern-Simons or BF theories. In the last one there appears an interesting interaction as the dynamic variables are coupled to their conjugate momenta via the covariant derivative.

1 Introduction

The non-linear σ model [1] is quite useful theory to describe a non-linear quantum dynamics. For example, the model is available as the low-energy effective theory of QCD, and gives us important knowledge of the physics, e.g. the soft pion physics. The model also gives us a powerful effective theory for describing macroscopic quantum phenomena, which appear in the condensed matter physics, e.g. the quantum Hall effect [2] or the high-T_c superconductivity [3]. The fruitfulness of the dynamics included in the model comes from the non-linearity [4–6], of course. At the same time, the non-linearity makes the analysis of the dynamics beyond the tree level very difficult [7].

In (d+1)-dimensions the model is defined by the Lagrangian density [5]:

$$\mathcal{L} = g^{d-1}\ tr(\partial_\mu U^\dagger \partial^\mu U) \quad , \tag{1}$$

where g is a coupling constant, which has the dimension of mass. U has its value on a group manifold G and tr is taken over the group index. Here, we consider the $SU(2)$ group as an example but the extension to other gauge groups is straightforward. In the usual formulation, U is written as $U = \exp{(i\varphi)}$ where φ is considered as a dynamic variable. The algebra-valued φ corresponds to a physical field like the pion in the case of QCD. The Lagrangian density (1) is expressed in a non-polynomial form concerned with φ field, so that we call this formulation the non-polynomial formulation. The Taylor expansion of (1) by φ gives us infinitely many types of interaction terms. It is very difficult to evaluate effects of radiative corrections in this way.

To overcome this difficulty, several types of formulations of the model are proposed and each of them has its own advantage. Sometimes a reformulation of the model gives us a new insight that is hard to find in the other formulation. One of these formulations, a polynomial representation of the non-linear σ model, has been studied extensively [8–12]. This formulation has many interesting features from the theoretical and practical points of view.

It is a specific feature of the polynomial formulation that the current density is regarded as the dynamic variable. The current density is a vector field which satisfies a *formal* flatness (or pure gauge) condition. The flatness condition is imposed as a constraint condition by introducing a Lagrange multiplier field. Then the model has *gauge symmetry* under a transformation of the Lagrange multiplier field. Thus, the Hamiltonian system of the model, so formulated, is a constrained system with the gauge symmetry. This situation motivates us to study the canonical structure of the model in detail.

Though the polynomial formulation is available in any space-time dimension, we restrict ourselves to consider the (2+1)-dimensional model in this paper. This is because the (2+1)-dimensional model is the simplest one, [1] which still has a first-class constraint making the canonical structure non-trivial. The model also can be applicable to a planar electron system in condensed matter physics.

[1] In the (1+1)-dimensional case which might be thought to be simpler, there is no local gauge symmetry [12]

In this paper we construct the generalized Hamiltonian formalism of the (2+1)-dimensional non-linear σ-model in the polynomial formulation explicitly by using the Dirac method [13] for constrained systems. In Sect. 2 we present a brief introduction to the polynomial formulation. The generalized Hamiltonian formalism of the model is extensively studied in Sect. 3. In Sect. 4 we derive three types of the pre-gauge-fixing Hamiltonian systems: In the first system, it is seen that the current algebra is realized as the fundamental Dirac Brackets. The second system has the Dirac brackets which are the same ones as the Chern-Simons or BF theories. In the third system there appears an interesting coupling that the dynamic variables are coupled to their conjugate momenta via the covariant derivative. Section 5 is devoted to conclusions and discussions.

2 Polynomial formulation

The basic idea of the polynomial formulation is as follows: We introduce an $SU(2)$ *algebra*-valued vector field L_μ such as $L_\mu = L^a_\mu \tau^a = g^{-\frac{1}{2}} U^\dagger \partial_\mu U$, where τ^a is the generator of the $SU(2)$ group. We consider the vector field L_μ as a dynamic variable and never refer to U. If we define the field strength $F^a_{\mu\nu}$ as $F^a_{\mu\nu} \equiv \partial_\mu L^a_\nu - \partial_\nu L^a_\mu + g^{\frac{1}{2}} f^{abc} L^b_\mu L^c_\nu$, where f^{abc} is the structure constant of the $SU(2)$ group, L^a_μ field satisfies the *flatness* or *pure gauge* condition, since $F^a_{\mu\nu} = 0$. Notice that the use of the term "flatness" is formal. While the original field U is transformed under the *global* $SU(2)$ group, the vector field L^a_μ is invariant under the transformation. Here, we impose the "flatness" condition to the theory as a *constraint condition* by introducing a Lagrange multiplier field denoted by $\theta^a_{\mu\nu}$. Then, we can obtain another description of the non-linear σ model, which is defined by the Lagrangian density:

$$\mathcal{L} = \frac{1}{2} g^2 L^a_\mu L^{a\mu} + \frac{1}{2} g \theta^{a\nu\rho} F^a_{\nu\rho} \quad . \tag{2}$$

The Lagrangian density (2) is the polynomial of L_μ and $\theta_{\mu\nu}$, so that we call this formulation the *polynomial* or first-order formulation of the non-linear σ-model. In our convention, [2] the mass dimensions of g, L^a_μ and θ^a_μ are 1, 1/2 and 1/2, respectively, and the generator τ^a is normalized by $tr(\tau^a \tau^b) = -(1/2)\delta^{ab}$.

Since $F^a_{\nu\rho}$ is antisymmetric under the interchange of ν and ρ, $\theta^a_{\nu\rho}$ also has to be. It is convenient to use the dual field θ^a_μ which is a vector field defined as $\theta^{a\nu\rho} = \epsilon^{\mu\nu\rho}\theta^a_\mu$. Then we can rewrite (2) as:

$$\mathcal{L} = \frac{1}{2} g^2 L^a_\mu L^{a\mu} + \frac{1}{2} g \epsilon^{\mu\nu\rho} \theta^a_\mu F^a_{\nu\rho} \quad , \tag{3}$$

which is our starting Lagrangian.

The Lagrangian density (3) has "*local* gauge symmetry" under a transformation of the Lagrange multiplier field θ^a_μ. The transformation is:

$$\theta^a_\mu \to \theta^a_\mu + D^{ab}_\mu \lambda^b, \tag{4}$$

[2] We have used the usual convention that the mass dimension of the vector field is 1/2 in (2+1)-dimensions

where $\lambda^b(x)$ is an arbitrary function. D^{ab}_μ is a covariant derivative defined as $D^{ab}_\mu \equiv \delta^{ab}\partial_\mu - g^{\frac{1}{2}} f^{abc} L^c_\mu$. We can easily show that the Lagrangian density (3) is invariant under the transformation (4) by using the Bianchi identity. The interesting feature of the transformation (4) is that it is the infinitesimal version of the non-Abelian gauge transformation. Thus, the symmetry is Abelian in fact [11]. The model in the polynomial formulation has the infinitesimal non-Abelian gauge symmetry as the exact symmetry.

The peculiar point of the polynomial formulation is that this property, which the dynamic variable should have, is imposed via the constraint. In addition, the constrained system has the local gauge symmetry under the transformation of the Lagrange multiplier. These situations make the structure of the Hamiltonian system nontrivial. In the following, we study the canonical structure of the system by using the Dirac method.

3 Dirac method

We construct the generalized Hamilton formalism by using the Dirac method [13] for constrained systems. We have to take care of the gauge symmetry under the transformation of θ_μ as (4). We may fix the gauge by adding any suitable gauge-fixing condition at the beginning, and then may quantize the gauge-fixed theory in the specific gauge. This procedure is too restrictive, because if we want to choose other gauge-fixing conditions, we have to repeat almost the same procedure again. Therefore, we use another prescription to make our result more general. We construct the pre-gauge-fixing Hamiltonian system (and get a set of the pre-gauge-fixing Dirac brackets), which maintains the gauge symmetry. Then, we can get a result that is independent of a choice of the gauge-fixing condition. After that, we may impose the remaining first-class constraints with any suitable gauge-fixing conditions on state vectors to restrict the phase space to a physical subspace, or we also can covert the first-class constraints to the second-class ones by adding the gauge-fixing conditions. If we construct the Dirac brackets, the second-class constraints become the strong equations. Once we construct the pre-gauge-fixing Hamiltonian system, we do not need to reconstruct everything from the beginning. The generalized Hamilton systems for the different gauge-fixing conditions can be obtained starting from the pre-gauge-fixing Hamiltonian system.

3.1 Primary system

The canonical momenta which are conjugate to the field variables L^a_μ and θ^a_μ are obtained as:

$$\pi^{a\mu} \equiv \frac{\delta L}{\delta \dot{L}^a_\mu} = g\epsilon^{0\mu\nu}\theta^a_\nu \quad , \tag{5}$$

$$\pi_\theta{}^{a\mu} \equiv \frac{\delta L}{\delta \dot{\theta}^a_\mu} = 0 \quad , \tag{6}$$

from (3) respectively, where $L = \int d\vec{x}\mathcal{L}$. Both of these equations do not include the first-order time-derivative term therefore they give us the primary constraints:

$$K^{a\mu} \equiv \pi^{a\mu} - g\epsilon^{0\mu\nu}\theta^a_\nu \approx 0 \quad , \tag{7}$$

$$\phi^{a\mu} \equiv \pi^{a\mu}_\theta \approx 0 \quad , \tag{8}$$

where "≈" means the weak equality as usual. The Poisson brackets between the field variables and their conjugate momenta are defined by:

$$\{L^a_\mu(t,\vec{x}), \pi^b_\nu(t,\vec{y})\} = g_{\mu\nu}\delta^{ab}\delta(\vec{x}-\vec{y}) \quad , \tag{9}$$

$$\{\theta^a_\mu(t,\vec{x}), \pi_{\theta}{}^b_\nu(t,\vec{y})\} = g_{\mu\nu}\delta^{ab}\delta(\vec{x}-\vec{y}) \quad . \tag{10}$$

The canonical Hamiltonian is obtained by the formal Legendre transformation as:

$$\begin{aligned} H_C &= \int d\vec{x}(\pi^a_\mu \dot{L}^{a\mu} + \pi^a_{\theta\mu}\dot{\theta^{a\mu}} - \mathcal{L}), \\ &= \int d\vec{x}(-g\epsilon^{0ij}\theta^a_0\partial_i L^a_j - g\epsilon^{0ij}\theta^a_i\partial_j L^a_0 - \frac{1}{2}g^2 L^a_\mu L^{a\mu} \\ &\quad -\frac{1}{2}g^{\frac{3}{2}}\epsilon^{\mu\nu\rho}f^{abc}\theta^a_\mu L^b_\nu L^c_\rho) \quad . \end{aligned} \tag{11}$$

In order to restrict the phase space by the primary constraints (7) and (8), we add the corresponding constraint terms to (11) which gives us the primary Hamiltonian:

$$H_P = H_C + \int d\vec{x}(u^a_\mu K^{a\mu} + v^a_\mu \phi^{a\mu}) \quad , \tag{12}$$

where $u^a_\mu(x)$ and $v^a_\mu(x)$ are the Lagrange multipliers.

3.2 Dirac algorithm

Starting from the primary constraints (7) and (8) and the primary Hamiltonian (12), we construct a consistent Hamiltonian system following the Dirac algorithm [13]. We require that a constraint f is not changed in time-evolution. It means that the constraint surface is independent of time, so that we can identify the true phase space definitely. Thus, we impose the consistency condition, $\dot{f} = \{f, H_P\} \approx 0$. As a result of this condition, three cases will be realized as follows: (i) A Lagrange multiplier might be determined. (ii) A new (secondary) constraint might be obtained. (iii) The condition might be satisfied consistently. In the case (ii), we impose again the consistency condition to the new constraint and repeat the same procedure until the case (i) or (iii) is realized. Finally, we will have a consistent set of constraints and at the same time some of the Lagrange multipliers will be determined. The resulting system is the generalized Hamiltonian system.

First, we impose the consistency condition to $K^{a\mu}$. For $\mu = 0$ there appears the secondary constraint:

$$M^a \equiv \epsilon^{0ij}\partial_i\theta^a_j + gL^{a0} + g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}\theta^b_i L^c_j \approx 0 \quad . \tag{13}$$

On the other hand, the consistency for K^{ai} determines the multiplier v^a_i as:

$$v^a_i = \partial_i\theta^a_0 - g\epsilon_{0ij}L^{aj} - g^{\frac{1}{2}}f^{abc}\theta^b_0 L^c_i + g^{\frac{1}{2}}f^{abc}\theta^b_i L^c_0 \quad . \tag{14}$$

Next, we require the consistency for $\phi^{a\mu}$. In the case of $\mu = 0$, we obtain the secondary constraint:

$$N^a \equiv \epsilon^{0ij}\partial_i L^a_j + \frac{1}{2}g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}L^b_i L^c_j \approx 0 \quad . \tag{15}$$

The consistency for ϕ^{ai} determines u^a_i to be:

$$u^a_i = \partial_i L^a_0 - g^{\frac{1}{2}}f^{abc}L^b_0 L^c_i \quad . \tag{16}$$

Now, we have two secondary constraints, (13) and (15). We repeat the same procedure for them. The consistency for M^a determines u^{a0} to be:

$$\begin{aligned} u^{a0} = -\frac{1}{g}(&g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}\theta^b_i u^c_j + \epsilon^{0ij}\partial_i v^a_j \\ &+g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}L^b_i v^c_j) \quad , \end{aligned} \tag{17}$$

where u^c_j and v^c_j have been given by (16) and (14), respectively. Finally, we impose the consistency condition to N^a and obtain the relation:

$$\epsilon^{0ij}\partial_i u^a_j + g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}L^b_i u^c_j \approx 0 \quad , \tag{18}$$

which can be rewritten as:

$$\epsilon^{0ij}f^{abc}L^b_0\partial_i L^c_j - g^{\frac{1}{2}}\epsilon^{0ij}L^b_0 L^b_i L^a_j \approx 0 \quad , \tag{19}$$

by substituting (16) for u^a_i in (18). We wonder if (19) gives us a new constraint, but we can show that the equation is satisfied automatically because of (15). Now we have completed finding all of the constraints, namely $K^{a\mu}$, $\phi^{a\mu}$, M^a and N^a.

3.3 Classification of constraints

The next step of the Dirac method is to classify these constraints into the first- and second-class constraints. The first-class constraint is the one which has the vanishing Poisson brackets with all of other constraints. The constraints except for the first-class ones are called the second-class constraints.

In order to simplify the notation, we define η^{as}'s as $\{\eta^{as}|a = 1,2,3; s = 1,..,8\} \equiv \{K^{a0}, K^{a1}, K^{a2}, \phi^{a0}, \phi^{a1}, \phi^{a2}, M^a, N^a\}$ where a is the group index. We define a matrix $B^{ab;st}(\vec{x},\vec{y})$ as $B^{ab;st}(\vec{x},\vec{y}) \equiv \{\eta^{as}(\vec{x}), \eta^{bt}(\vec{y})\}$. Then we have:

$$\begin{aligned} &B^{ab;st}(\vec{x},\vec{y}) = \\ &\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -g\delta^{ab} & 0 \\ 0 & 0 & 0 & 0 & 0 & -g\delta^{ab} & g^{\frac{1}{2}}f^{abc}\theta^c_2 & -D^{ab}_2(\vec{x}) \\ 0 & 0 & 0 & 0 & g\delta^{ab} & 0 & -g^{\frac{1}{2}}f^{abc}\theta^c_1 & D^{ab}_1(\vec{x}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g\delta^{ab} & 0 & 0 & 0 & -D^{ab}_2(\vec{x}) & 0 \\ 0 & g\delta^{ab} & 0 & 0 & 0 & 0 & D^{ab}_1(\vec{x}) & 0 \\ g\delta^{ab} & g^{\frac{1}{2}}f^{abc}\theta^c_2 & -g^{\frac{1}{2}}f^{abc}\theta^c_1 & 0 & -D^{ab}_2(\vec{x}) & D^{ab}_1(\vec{x}) & 0 & 0 \\ 0 & -D^{ab}_2(\vec{x}) & D^{ab}_1(\vec{x}) & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\times\delta(\vec{x}-\vec{y}), \end{aligned} \tag{20}$$

by calculating the Poisson brackets between η^{as}'s. In (20) we find that the ϕ^{a0} is the first-class constraint and the others are the second-class ones. Thus, we have one first-class constraint and seven second-class constraints.[3] The number of the second-class constraints should be even, because a coordinate should make a pair with a momentum and the dimensions of the reduced phase space should be even. We have now the seven second-class constrains. Therefore, we can get at least one more first-class constraint by a linear combination of the second-class constraints. This means that these second-class constraints are not linearly independent. We may convert N^a to the first-class constraint. For convenience we denote the second-class constraints by ξ^{as}'s as $\{\xi^{as}|a=1,2,3;s=1,..,6\}=\{K^{a0},K^{a1},K^{a2},\phi^{a1},\phi^{a2},M^a\}$. By using ξ^{as}'s, we define a matrix $C^{ab;st}$ as $C^{ab;st}(\vec{z}_1,\vec{z}_2)\equiv\{\xi^{as}(\vec{z}_1),\xi^{bt}(\vec{z}_2)\}$. Then the new first-class constraint is defined as:

$$G^a(\vec{x})=N^a(\vec{x})-\int d\vec{z}_1 d\vec{z}_2\{N^a(\vec{x}),\xi^{bs}(\vec{z}_1)\}\times(C^{-1})^{bc}_{st}(\vec{z}_1,\vec{z}_2)\xi^{ct}(\vec{z}_2), \tag{21}$$

which is actually the linear combination of N^a and ξ^{as}'s. $(C^{-1})^{ab}_{st}$ is the inverse matrix of $C^{ab;st}$. We can easily check that the Poisson brackets between G^a and all of other constraints vanish, so that G^a is surely the first-class constraint. Because the determinant of $C^{ab;st}$ is not zero, the remaining second-class constraints are independent of each other and $(C^{-1})^{as}_{st}$ exists certainly. There are no more first-class constraints.

The explicit form of the matrix $C^{ab;st}$ is obtained after deleting two rows and two columns, which are concerned with ϕ^{a0} and N^a, from the matrix $B^{ab;st}$. Thus, we have:

$$C^{ab;st}(\vec{z}_1,\vec{z}_2)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -g\delta^{ab} \\ 0 & 0 & 0 & 0 & -g\delta^{ab} & g^{\frac{1}{2}}f^{abc}\theta^c_2 \\ 0 & 0 & 0 & g\delta^{ab} & 0 & -g^{\frac{1}{2}}f^{abc}\theta^c_1 \\ 0 & 0 & -g\delta^{ab} & 0 & 0 & -D^{ab}_2(\vec{z}_1) \\ 0 & g\delta^{ab} & 0 & 0 & 0 & D^{ab}_1(\vec{z}_1) \\ g\delta^{ab} & g^{\frac{1}{2}}f^{abc}\theta^c_2 & -g^{\frac{1}{2}}f^{abc}\theta^c_1 & -D^{ab}_2(\vec{z}_1) & D^{ab}_1(\vec{z}_1) & 0 \end{pmatrix}\times\delta(\vec{z}_1-\vec{z}_2). \tag{22}$$

The inverse of $C^{ab;st}$ also can be evaluated explicitly which becomes:

$$(C^{-1})^{ab}_{st}(\vec{z}_1,\vec{z}_2)=\begin{pmatrix} Q^{ab}(\vec{z}_1,\vec{z}_2) & \frac{1}{g^2}D^{ab}_1(\vec{z}_1) & \frac{1}{g^2}D^{ab}_2(\vec{z}_1) & -\frac{1}{g^{\frac{3}{2}}}f^{abc}\theta^c_1 & -\frac{1}{g^{\frac{3}{2}}}f^{abc}\theta^c_2 & \frac{1}{g}\delta^{ab} \\ \frac{1}{g^2}D^{ab}_1(\vec{z}_1) & 0 & 0 & 0 & \frac{1}{g}\delta^{ab} & 0 \\ \frac{1}{g^2}D^{ab}_2(\vec{z}_1) & 0 & 0 & -\frac{1}{g}\delta^{ab} & 0 & 0 \\ -\frac{1}{g^{\frac{3}{2}}}f^{abc}\theta^c_1 & 0 & \frac{1}{g}\delta^{ab} & 0 & 0 & 0 \\ -\frac{1}{g^{\frac{3}{2}}}f^{abc}\theta^c_2 & -\frac{1}{g}\delta^{ab} & 0 & 0 & 0 & 0 \\ -\frac{1}{g}\delta^{ab} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\times\delta(\vec{z}_1-\vec{z}_2), \tag{23}$$

[3] The number of constraints is counted without distinguishing the group index

where we have defined the operator $Q^{ab}(\vec{z}_1,\vec{z}_2)$ as:

$$\begin{aligned} Q^{ab}(\vec{z}_1,\vec{z}_2)\delta(\vec{z}_1-\vec{z}_2) &= \frac{1}{g^{\frac{5}{2}}}\epsilon^{0ij}\{f^{acd}\theta^d_i(\vec{z}_1)D^{cb}_j(\vec{z}_1) \\ &\quad +f^{cbd}D^{ac}_i(\vec{z}_1)\theta^d_j(\vec{z}_2)\}\delta(\vec{z}_1-\vec{z}_2), \\ &= \frac{1}{g^{\frac{5}{2}}}\epsilon^{0ij}f^{abc}(D^{cd}_i(\vec{z}_1)\theta^d_j(\vec{z}_1)) \\ &\quad \times\delta(\vec{z}_1-\vec{z}_2) \ . \end{aligned} \tag{24}$$

By using (15), (21) and (23), we can derive the explicit form of $G^a(\vec{x})$ as:

$$\begin{aligned} G^a(\vec{x}) &= N^a(\vec{x})-\frac{1}{g^2}\epsilon^{0ij}D^{ab}_i(\vec{x})D^{bc}_j(\vec{x})K^{c0}(\vec{x}) \\ &\quad +\frac{1}{g}D^{ab}_i(\vec{x})\phi^{bi}(\vec{x}) \ . \end{aligned} \tag{25}$$

3.4 Dirac brackets

Now we have two first-class constraints, ϕ^{a0} and G^a, and the six second-class constraints ξ^{as}'s. We construct the Dirac brackets allowing us to use the second-class constraints as equality relations, which are called "strong equations". Thus, the Dirac brackets give us the canonical algebra on the constrained phase space. The definition of the Dirac brackets is:

$$\begin{aligned} \{A(\vec{x}),B(\vec{y})\}_D &\equiv \{A(\vec{x}),B(\vec{y})\} \\ &\quad -\int d\vec{z}_1 d\vec{z}_2\{A(\vec{x}),\xi^{as}(\vec{z}_1)\} \\ &\quad \times(C^{-1})^{ab}_{st}(\vec{z}_1,\vec{z}_2)\{\xi^{bt}(\vec{z}_2),B(\vec{y})\} \end{aligned} \tag{26}$$

for any variable $A(\vec{x})$ and $B(\vec{y})$. After tedious but straightforward calculations, we find the following Dirac brackets:

$$\{L^a_0(\vec{x}),\pi^b_i(\vec{y})\}_D=-\frac{1}{g^{\frac{1}{2}}}\epsilon_{0ij}f^{abc}\theta^{cj}\delta(\vec{x}-\vec{y}) \ , \tag{27}$$

$$\{L^a_i(\vec{x}),\pi^b_j(\vec{y})\}_D=g_{ij}\delta^{ab}\delta(\vec{x}-\vec{y}) \ , \tag{28}$$

$$\{L^a_0(\vec{x}),L^b_0(\vec{y})\}_D=Q^{ab}(\vec{x},\vec{y})\delta(\vec{x}-\vec{y}) \ , \tag{29}$$

$$\{L^a_0(\vec{x}),L^b_i(\vec{y})\}_D=\frac{1}{g^2}D^{ab}_i(\vec{x})\delta(\vec{x}-\vec{y}) \ , \tag{30}$$

$$\{\theta^a_0(\vec{x}),\pi_{\theta 0}^b(\vec{y})\}_D=\delta^{ab}\delta(\vec{x}-\vec{y}) \ , \tag{31}$$

$$\{L^a_0(\vec{x}),\theta^b_i(\vec{y})\}_D=-\frac{1}{g^{\frac{3}{2}}}f^{abc}\theta^c_i\delta(\vec{x}-\vec{y}) \ , \tag{32}$$

$$\{L^a_i(\vec{x}),\theta^b_j(\vec{y})\}_D=\frac{1}{g}\delta^{ab}\epsilon_{0ij}\delta(\vec{x}-\vec{y}) \ . \tag{33}$$

The other Dirac brackets vanish. (See the Appendix.)

Under the use of these brackets, the second-class constraints can be regarded as the strong equations. Summarizing them here, we have the strong equations as follows:

$$K^{a0}=\pi^{a0}=0 \ , \tag{34}$$

$$K^{ai}=\pi^{ai}-g\epsilon^{0ij}\theta^a_j=0 \ , \tag{35}$$

$$\phi^{ai}={\pi_\theta}^{ai}=0 \ , \tag{36}$$

$$M^a=\epsilon^{0ij}\partial_i\theta^a_j+gL^{a0}+g^{\frac{1}{2}}\epsilon^{0ij}f^{abc}\theta^b_iL^c_j=0 \ . \tag{37}$$

On the other hand, we have two first-class constraints ϕ^{a0} and G^a. It should be noticed that G^a is reduced to N^a if we use the strong equations. Eventually the first-class constraints become:

$$\phi^{a0} = \pi_\theta^{a0} \approx 0 \quad , \tag{38}$$

$$G^a = \epsilon^{0ij} F_{ij}^a \approx 0 \quad . \tag{39}$$

Notice that (39) is the Gauss law constraint which is the same one that appeared in the Chern-Simons theory [14] or the BF theory [15]. It has its origin in the term $\epsilon^{\mu\nu\rho}\theta_\mu F_{\nu\rho}$, which is a topological term independent of the metric as is the Chern-Simons term. Therefore, the Hamiltonian system may be considered to share a common nature with the ones of these theories.

3.5 Remarks on quantization

Once we obtain the generalized Hamiltonian system, the procedure of quantization is almost straightforward. Here, we just give general remarks on the quantization.

In the generalized Hamiltonian system which we have obtained, two first-class constraints have appeared. In treating the first-class constraints, there are two strategies as follows:

(a) We replace the Dirac brackets to the commutation relations. Thus, we introduce the quantum operators first. Then, we impose the first-class constraints and also any suitable gauge-fixing conditions on the state vectors in the Hilbert space and obtain the true phase space.

(b) We may fix the gauge symmetry by imposing any suitable gauge-fixing conditions. Because we have non-vanishing Poisson brackets between the first-class constraints and the gauge-fixing conditions, the first-class constraints are regarded as the second-class constraints. We can construct the gauge-fixed Dirac brackets which allow us to use all constraints and the gauge-fixing conditions as the strong equations. Then, the system is quantized by replacing the gauge-fixed Dirac brackets to the commutation relations.

We also have to be careful to order the quantum operators as usual. In this aspect too, the polynomial formulation is much better than the non-polynomial one, because we treat only the polynomial not the infinite power of the operators.

We may rely on the path-integral quantization method too. The Faddeev-Senjanovic method [16] gives us a systematic prescription of the path-integral quantization. For the path-integral quantization we need the generalized Hamiltonian formalism as the strict basis to determine the path-integral measure.

4 Pre-gauge-fixing Hamiltonian systems

The pre-gauge-fixing total Hamiltonian H_T is defined by adding the constraint term due to the secondary first-class constraint G^a to the primary Hamiltonian H_P given in (12). Thus, H_T becomes:

$$H_T = H_P + \int d\vec{x} w^a G^a \quad , \tag{40}$$

where $w^a(x)$ is the Lagrange multiplier. The field variables which are used in the starting Lagrangian density (3), are L^0, L^{ai}, π^{a0}, π^{ai}, θ^{a0}, θ^{ai}, π_θ^{a0} and π_θ^{ai}. In terms of the strong equations (34) $\sim$ (37), we can eliminate some of them from the total Hamiltonian H_T. We also may retain some redundant variables in H_T with some strong equations. The explicit form of H_T depends on which variables are eliminated.

4.1 H_T by L_μ^a and current algebra

H_T obtained most naively is:

$$\begin{aligned} H_T = \int d\vec{x}(&\frac{1}{2}g^2 L_0^a L^{a0} - \frac{1}{2}g^2 L_i^a L^{ai} \\ &+ w^a G^a + v_0^a \phi^{a0}) \quad , \end{aligned} \tag{41}$$

where we have shifted $w^a - \frac{1}{2}g\theta_0^a$ to w^a. The last two terms in (41) correspond to the first-class constraints. In this form all of L_μ^a's are kept as dynamic variables.

The current algebra is one of most important features of the non-linear σ model. In the polynomial formulation, we have considered the current density as the dynamic variable. We do not refer to any elementary fields, of which the current density is composed. The current algebra should be realized as the fundamental Dirac brackets.

To see that, let us consider the Dirac brackets of (29) and (30). By using (24) and (37), we can rewrite (29) as:

$$\{L^{a0}(\vec{x}), L^{b0}(\vec{y})\}_D = -\frac{1}{g^{\frac{3}{2}}} f^{abc} L^{c0}(\vec{x})\delta(\vec{x}-\vec{y}) \quad . \tag{42}$$

This is the well-known form of the Lie algebra. From (30) we also have:

$$\begin{aligned} \{L^{a0}(\vec{x}), L^{bi}(\vec{y})\}_D = &-\frac{1}{g^{\frac{3}{2}}} f^{abc} L_i^c(\vec{x})\delta(\vec{x}-\vec{y}) \\ &+\frac{1}{g^2}\delta^{ab}\partial_i^x \delta(\vec{x}-\vec{y}) \quad . \end{aligned} \tag{43}$$

It should be noticed that the second term in the right-hand side of (43) is the so-called Schwinger term [17]. The algebra is consistent with the general form of the current algebra expected in this kind of the model. [18]

4.2 H_T by L_i^a and θ_i^a

We can express H_T by L_i^a and θ_i^a. To this end, we eliminate L_0^a from (41) by using:

$$L^{a0} = -\frac{1}{g}\epsilon^{0ij} D_i^{ab}\theta_j^b \quad , \tag{44}$$

which is obtained from (37). Then we obtain:

$$H_T = \int d\vec{x}\{\frac{1}{2}(\epsilon^{0ij} D_i^{ab}\theta_j^b)(\epsilon^{0kl} D_k^{ac}\theta_l^c) \\ -\frac{1}{2}g^2 L_i^a L^{ai} + w^a G^a + v_0^a \phi^{a0}\} \quad . \qquad (45)$$

The Dirac brackets for L_i^a and θ_i^a are given by (33). The brackets have a characteristic form including ϵ^{0ij} that is same as the ones of the Chern-Simons or BF theories. This is expected because the Lagrangian density (3) includes the symplectic form as $\epsilon^{\mu\nu\rho}\theta_\mu F_{\nu\rho}$.

4.3 H_T by L_i^a and π_i^a

We also can obtain the expression of H_T in which the canonical pairs of coordinates and momenta can be seen explicitly. Let us rewrite (35) as:

$$\epsilon^{0ij}\theta_j^a = \frac{1}{g}\pi^{ai} \quad . \qquad (46)$$

Notice that π_i^a is a dual field of θ_j^a. In eliminating θ_j^b in (45) by (46), we obtain:

$$H_T = \int d\vec{x}\{\frac{1}{2g^2}(D_i^{ab}\pi^{bi})(D_j^{ac}\pi^{cj}) \\ -\frac{1}{2}g^2 L_i^a L^{ai} + w^a G^a + v_0^a \phi^{a0}\} \quad . \qquad (47)$$

Finally, the Hamiltonian (47) has been obtained by using all of the strong equations (34) $\sim$ (37). The Dirac brackets (28) shows that L^{ai} and π^{ai} (i=1, 2) make two canonical pairs. ϕ^{a0} includes π_θ^{a0} so that one more pair (θ^{a0}, π_θ^{a0}) exists. Thus, we have three canonical pairs in the system. Since we have two first-class constraints, two gauge-fixing conditions are needed to reduce the phase space to the true one. This means that the true degrees of freedom in the model is just one (counting the number of the canonical pairs).

The specific feature seen in (47) is that L_i^a's are coupled to π_i^a's through the covariant derivative D_i^{ab}. It is interesting that a dynamic variable is coupled to its conjugate momentum by way of the minimal coupling.

5 Conclusions and discussions

We have constructed the generalized Hamiltonian formalism of the (2+1)-dimensional non-linear σ model in polynomial formulation by using the Dirac method for constrained systems. In the polynomial formulation the current density is considered as the dynamic variable. The current density satisfies the flatness condition that is imposed on the Hamiltonian system as the constraint by introducing the Lagrange multiplier field. Following the Dirac algorithm, we have derived the full set of constraints which have been classified into first- and second-class constraints. Since the system is symmetric under the local gauge transformation of the Lagrangian multiplier field, the first-class constraint which corresponds to the Gauss law has appeared as the secondary constraint. We have evaluated the Dirac brackets that allow us to use the second-class constraints as the strong equations. Reducing the variables by the strong equations, we have found the pre-gauge-fixing total Hamiltonian systems.

The explicit form of the pre-gauge-fixing total Hamiltonian depends on which the variables are eliminated by the strong equations. As typical ones, three types of the Hamiltonian systems have been derived:

1) The first type is suitable for discussing the current algebra. We have reproduced the correct current algebra as the Dirac brackets. It is remarkable that the current algebra is realized as the fundamental Dirac brackets without referring to the variables of which the current density is composed. We may say that the polynomial formulation gives us a concrete canonical formalism of the Sugawara theory [19].
2) The second type has the Dirac brackets which are similar to the ones of the Chern-Simons or BF theories. This is because the model in the polynomial formulation has the symplectic form as $\epsilon^{\mu\nu\rho}\theta_\mu F_{\nu\rho}$, which is added as the constraint term imposing the flatness condition to the model. It is interesting to study how the polynomial formulation is related to these theories.
3) In the third type the minimal set of the canonical pairs appears. In fact, the number of the true degree of freedom is just one in counting the number of the canonical pairs. The interesting aspect of this type is that the dynamic variables are coupled to their conjugate momenta via the covariant derivative. This kind of interaction is not so familiar. In addition, we may say that this type is dual to the second type as seen in (46). It may be important to understand the meaning of the duality.

We should observe that these Hamiltonian systems, each of which has the remarkable characteristics as mentioned above, are derived from the same Lagrangian density. It may be important that we know how these systems are converted from one to another by changing the dynamic variables. Each of the Hamiltonian systems is a different representation of the same model. This has been clarified by constructing the *pre-gauge-fixing* Hamiltonian systems.

One of our aims using the polynomial formulation is to evaluate the radiative corrections by the quantized fields. The generalized Hamiltonian formalism obtained in this paper gives us the basis for the quantization of the model in the polynomial formulation. Then, the concrete method to proceed with these evaluations may be perturbation, for instance. In [11] the perturbation of the model in the polynomial formulation under the covariant gauge-fixing condition has been given, where we have found a new perturbative series in which it is expected that the ultraviolet divergence is much milder than the one in the non-polynomial formulation. The Hamiltonian system used there corresponds to the third type obtained here. It may be interesting to study the perturbation theories based on the first or second type Hamiltonian systems, which are under construction.

Appendix

We present here some results of calculating Poisson brackets between the dynamic variables (L^a_μ, π^a_μ, θ^a_μ, and $\pi^a_{\theta\mu}$) and the second-class constraints ξ^{as}'s ($\{\xi^{as}|a=1,2,3; s=1,..,6\} = \{K^{a0}, K^{a1}, K^{a2}, \phi^{a1}, \phi^{a2}, M^a\}$), which are needed for deriving the Dirac brackets in Sect. 3.4. For example, let us consider the Poisson brackets between L^a_μ's and ξ^{as}'s. We have:

$$\begin{aligned}\{L^a_\mu(\vec{x}), K^{b0}(\vec{z}_1)\} &= g^0_\mu \delta^{ab}\delta(\vec{x}-\vec{z}_1) \quad ,\\ \{L^a_\mu(\vec{x}), K^{b1}(\vec{z}_1)\} &= g^1_\mu \delta^{ab}\delta(\vec{x}-\vec{z}_1) \quad ,\\ \{L^a_\mu(\vec{x}), K^{b2}(\vec{z}_1)\} &= g^2_\mu \delta^{ab}\delta(\vec{x}-\vec{z}_1) \quad ,\end{aligned}$$

where the other brackets vanish. All of them are collected in a formula as:

$$\{L^a_\mu(\vec{x}), \xi^{bs}(\vec{z}_1)\} = (g^0_\mu\delta^{ab}, g^1_\mu\delta^{ab}, g^2_\mu\delta^{ab}, 0, 0, 0)\delta(\vec{x}-\vec{z}_1) \quad .$$

In the same way, we obtain:

$$\begin{aligned}\{\pi^a_\mu(\vec{x}), \xi^{bs}(\vec{z}_1)\} &= (0,0,0,0,0,-gg^0_\mu\delta^{ab}\\ &\quad -g^{\frac{1}{2}}\epsilon^{0ij}g_{\mu j}f^{abc}\theta^c_i)\delta(\vec{x}-\vec{z}_1) \quad ,\\ \{\theta^a_\mu(\vec{x}), \xi^{bs}(\vec{z}_1)\} &= (0,0,0,g^1_\mu\delta^{ab}, g^2_\mu\delta^{ab}, 0)\delta(\vec{x}-\vec{z}_1) \quad ,\\ \{\pi^a_{\theta\mu}(\vec{x}), \xi^{bs}(\vec{z}_1)\} &= (0, gg_{\mu 2}\delta^{ab}, -gg_{\mu 1}\delta^{ab}, 0, 0,\\ &\quad -\epsilon^{0ij}g_{\mu j}\delta^{ab}\partial^{z_1}_i + g^{\frac{1}{2}}\epsilon^{0ij}g_{\mu i}f^{abc}L^c_j)\\ &\quad \times\delta(\vec{x}-\vec{z}_1),\\ \{\xi^{ct}(\vec{z}_2), L^d_\nu(\vec{y})\} &= (-g^0_\nu\delta^{cd}, -g^1_\nu\delta^{cd}, -g^2_\nu\delta^{cd}, 0, 0, 0)\\ &\quad \times\delta(\vec{z}_2-\vec{y}) \quad ,\\ \{\xi^{ct}(\vec{z}_2), \pi^d_\nu(\vec{y})\} &= (0,0,0,0,0,gg^0_\nu\delta^{cd}\\ &\quad -g^{\frac{1}{2}}\epsilon^{0ij}g_{j\nu}f^{cde}\theta^e_i)\delta(\vec{z}_2-\vec{y}) \quad ,\\ \{\xi^{ct}(\vec{z}_2), \theta^d_\nu(\vec{y})\} &= (0,0,0,-g^1_\nu\delta^{cd}, -g^2_\nu\delta^{cd}, 0)\delta(\vec{z}_2-\vec{y}) \quad ,\\ \{\xi^{ct}(\vec{z}_2), \pi^d_{\theta\nu}(\vec{y})\} &= (0, -gg_{2\nu}\delta^{cd}, gg_{1\nu}\delta^{cd},\\ &\quad 0, 0, \epsilon^{0ij}g_{j\nu}\delta^{cd}\partial^{z_2}_i\\ &\quad +g^{\frac{1}{2}}\epsilon^{0ij}g_{i\nu}f^{cde}L^e_j)\delta(\vec{z}_2-\vec{y}) \quad .\end{aligned}$$

Using these formulae with (26), the Dirac brackets (27) $\sim$ (33) are obtained.

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